

RESULTS ON WEIGHTED NORM INEQUALITIES FOR MULTIPLIERS

BY

DOUGLAS S. KURTZ AND RICHARD L. WHEEDEN¹

ABSTRACT. Weighted L^p -norm inequalities are derived for multiplier operators on Euclidean space. The multipliers are assumed to satisfy conditions of the Hörmander-Mikhlin type, and the weight functions are generally required to satisfy conditions more restrictive than A_p , which depend on the degree of differentiability of the multiplier. For weights which are powers of $|x|$, sharp results are obtained which indicate such restrictions are necessary. The method of proof is based on the function $f^\sharp$ of C. Fefferman and E. Stein rather than on Littlewood-Paley theory. The method also yields results for singular integral operators.

1. Let $m(x)$ be a bounded function on $\mathbb{R}^n$ and consider the multiplier operator Tf defined initially for functions f in the Schwartz space $\mathcal{S}$ by $(Tf)^\wedge(x) = m(x)\hat{f}(x)$, where $\hat{g}$ is the Fourier transform of g . Denote by s a real number greater than or equal to 1, l a positive integer, and $\alpha = (\alpha_1, \dots, \alpha_n)$ a multi-index of nonnegative integers α_j with length $|\alpha| = \alpha_1 + \dots + \alpha_n$. We say $m \in M(s, l)$ if

$$\sup_{R>0} \left(R^{s|\alpha|-n} \int_{R<|x|<2R} |D^\alpha m(x)|^s dx \right)^{1/s} < +\infty \quad \text{for all } |\alpha| \leq l. \quad (1.1)$$

The condition (1.1) has been known to be related to multiplier theorems for some time. The classic works in this direction are the theorems of Marcinkiewicz (see [18]) and Hörmander-Mikhlin (see [7]):

THEOREM A. Let $n = 1$, $1 < p < \infty$, and $m \in M(1, 1)$. Then there exists a constant C , independent of f , such that $\|Tf\|_p \leq C\|f\|_p$.

THEOREM B. Let $l > n/2$, $1 < p < \infty$, and $m \in M(2, l)$. Then there exists a constant C , independent of f , such that $\|Tf\|_p \leq C\|f\|_p$.

Much work has been done to extend these results. Using interpolation methods, Calderón and Torchinsky [2] have considered the condition $m \in M(s, l)$ for $s \geq 2$ and $l > n/s$. Hirschman [6], Krée [11], and Triebel [20] have extended these results in various directions to weighted L^p spaces for weights which are powers of $|x|$. More recently, Kurtz [12] extended Theo-

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rems A and B to L^p spaces with more general weights by using the weighted norm inequalities derived in [15] for the function g_λ^* .

The purpose of this paper is two-fold. We consider $s \leq 2$ and present a method of proof based on the function $f^\#$ of Fefferman and Stein [5] rather than on Littlewood-Paley theory.

We say $f \in L_w^p(\mathbb{R}^n)$, $1 < p < \infty$ and $w(x) > 0$, if

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < +\infty.$$

The weights w we will consider satisfy an A_r condition; i.e., $w \in A_r$ if there is a constant C such that

$$\left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(r-1)} dx \right)^{r-1} \leq C, \quad 1 < r < \infty,$$

$$\frac{1}{|Q|} \int_Q w(x) dx \leq C \operatorname{ess\,inf}_Q w, \quad r = 1,$$

for all cubes $Q \subset \mathbb{R}^n$. When $r = 1$, the condition that $w \in A_1$ means $w^*(x) \leq Cw(x)$ for almost every x , where g^* is the Hardy-Littlewood maximal function of g . Finally, $w \in A_\infty$ if there exist positive constants C and δ such that for any cube $Q \subset \mathbb{R}^n$ and for any measurable set $E \subset Q$,

$$\frac{m_w(E)}{m_w(Q)} \leq C \left(\frac{|E|}{|Q|} \right)^\delta,$$

where $m_w(E) = \int_E w(x) dx$. Results concerning A_p functions can be found in Muckenhoupt [13] and Coifman and Fefferman [3]. Note, in particular, that $w \in A_p$ implies $w \in A_\infty$.

We use p' to denote the index conjugate to p : $1/p + 1/p' = 1$, $p > 1$.

The main result of this paper is:

THEOREM 1. *Let $1 < s \leq 2$, $n/s < l \leq n$, and $m \in M(s, l)$. If*

(1) $n/l < p < \infty$ and $w \in A_{pl/n}$, or

(2) $1 < p < (n/l)'$ and $w^{-1/(p-1)} \in A_{p'l/n}$,

then there is a constant C , independent of f , such that

$$\|Tf\|_{p,w} \leq C\|f\|_{p,w}.$$

When $l < n$, we may take $p = n/l$ in (1) and $p = (n/l)'$ in (2). If

(3) $w^{n/l} \in A_1$,

there is a constant C , independent of f and λ , such that

$$m_w(\{x \in \mathbb{R}^n: |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{1,w}, \quad \lambda > 0.$$

Using interpolation, other conditions on the weight can be found which guarantee that T is a bounded operator. One result which we will prove is:

THEOREM 2. *If $1 < p < \infty$, $1 < s < 2$, $n/s < l < n$, $m \in M(s, l)$, and $w^{n/l} \in A_p$ then*

$$\|Tf\|_{p,w} \leq C\|f\|_{p,w}$$

for a constant independent of f .

This result does not give the best possible condition on the weight. When $w(x) = |x|^\beta$, we have $w \in A_p$ if $-n < \beta < n(p-1)$. Interpreting Theorem 1 for such w and using interpolation with change of measures, we will show:

THEOREM 3. *Let $1 < s < 2$, $n/s < l < n$, and $m \in M(s, l)$. If $1 < p < \infty$ and $\max\{-n, -lp\} < \beta < \min\{n(p-1), lp\}$, then there is a constant C , independent of f , such that*

$$\|Tf\|_{p,|x|^\beta} \leq C\|f\|_{p,|x|^\beta}.$$

In particular, if $n/l < p < (n/l)'$, we get $-n < \beta < n(p-1)$; we may also take $p = n/l$ and $p = (n/l)'$ if $l < n$.

We will show that this result is sharp with the possible exception of the endpoint values of β .

Let $\check{g}$ denote the inverse Fourier transform of g . If we set $K = \check{m}$, then for $f \in \mathcal{S}$, $Tf(x) = (K * f)(x)$. Our proof of Theorem 1 is based on using information about m to get estimates on approximations to K , so it is not surprising that the technique carries over to convolution operators.

Denote by $\Sigma = \Sigma_{n-1} = \{x \in \mathbb{R}^n: |x| = 1\}$, $x' = x/|x| \in \Sigma$ ($x \neq 0$), and ρ any rotation of Σ with magnitude $|\rho| = \sup_{x \in \Sigma} |\rho x - x|$. Let $1 < r < \infty$ and $\Omega \in L^r(\Sigma)$ be positively homogeneous of degree zero. We say that Ω satisfies the L^r -Dini condition if

$$\int_0^1 \omega_r(\delta) \frac{d\delta}{\delta} < +\infty,$$

where

$$\omega_r(\delta) = \sup_{|\rho| < \delta} \left(\int_{\Sigma} |\Omega(\rho x) - \Omega(x)|^r d\sigma_x \right)^{1/r}.$$

Set $K(x) = \Omega(x')/|x|^n$, with $\int_{\Sigma} \Omega(x) d\sigma_x = 0$, and $Tf(x) = (K * f)(x)$ in the usual principal-value sense. If Ω satisfies the L^r -Dini condition then it also satisfies the L^1 -Dini condition, which by [1] implies T is a bounded operator on L^p , $1 < p < \infty$. Recently, Kaneko and Yano [10] have shown that if Ω satisfies the L^∞ -Dini condition then T maps L_w^p into itself for $1 < p < \infty$ and $w \in A_p$. We have extended this to:

THEOREM 4. *Let $1 < r < \infty$, $\Omega \in L^r(\Sigma)$, and $\int_{\Sigma} \Omega(x) d\sigma_x = 0$. Suppose Ω satisfies the L^r -Dini condition. If*

(1) $r' < p < \infty$ and $w \in A_{p/r'}$, or

(2) $1 < p < r$ and $w^{-1/(p-1)} \in A_{p'/r}$
 then there is a constant C , independent of f , such that

$$\|Tf\|_{p,w} \leq C\|f\|_{p,w}.$$

When $r < \infty$, we may take $p = r'$ in (1) and $p = r$ in (2). If

(3) $w^{r'} \in A_1$, then

$$m_w(\{x \in \mathbb{R}^n: |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{1,w}, \quad \lambda > 0,$$

where C is independent of f and λ .

Theorem 4 is a direct analogue of Theorem 1. (We could also have stated a version of Theorem 3. See also [14].) In fact, when $r > 2$, r' plays the same role as n/l . For example, notice the similarity between $m \in M(s, n)$, $1 < s < 2$, and Ω satisfying the L^∞ -Dini condition. Our technique, however, does not allow for either r or s to be equal to 1.

§2 contains the basic lemma and a collection of results used in the proof of Theorem 1. This theorem and Theorems 2 and 3 are proved in §3. The proof of Theorem 4 is found in §4. The paper concludes with a counterexample showing Theorem 3 is best possible except for the question of endpoint equalities for β . The basic lemma and the counterexample are generalizations to $n > 1$ of results in [16], and we gratefully acknowledge many helpful discussions with W.-S. Young and B. Muckenhoupt.

2. Following [7], we select an approximation to the identity

$$\sum_{j=-\infty}^{+\infty} \phi(2^{-j}x) = 1, \quad x \neq 0,$$

where ϕ is an infinitely differentiable, nonnegative function supported in $\frac{1}{2} < |x| < 2$. Let $m_j(x) = m(x)\phi(2^{-j}x)$, so that

$$m(x) = \sum_{j=-\infty}^{+\infty} m_j(x), \quad x \neq 0.$$

Notice that $m_j(x)$ is supported in $2^{j-1} < |x| < 2^{j+1}$ and that for such x , $m_k(x) = 0$ unless $k = j - 1, j$, or $j + 1$. It follows easily that if $m \in M(s, l)$ and $|\alpha| < l$, then

$$\left(\int_{\mathbb{R}^n} |D^\alpha m_j(x)|^s dx \right)^{1/s} \leq C(2^j)^{n/s-|\alpha|},$$

with C independent of j .

We also have that $m_j \in L^1 \cap L^\infty$. Define $k_j(x)$ by $k_j(x) = \check{m}_j(x)$, and let

$$m^N(x) = \sum_{j=-N}^N m_j(x), \quad K_N(x) = (m^N)^\sim(x) = \sum_{j=-N}^N k_j(x).$$

It follows that $\|m^N\|_\infty \leq C$, uniformly in N , and that $m^N(x) \rightarrow m(x)$, $x \neq 0$,

as $N \rightarrow \infty$. Now define $T_N f$ by $T_N f = (m^N \hat{f})^\vee$, so that $T_N f = f * K_N$ for $f \in L^2$, say. The following lemma shows how conditions on m can be interpreted as conditions on K_N .

LEMMA 1. Let $1 < s < 2$, $m \in M(s, l)$ for a positive integer l , and let K_N be defined as above. If d is an integer such that $0 < d \leq l$, $1 < t \leq s$, $n/t < d < n/t + 1$, and $1 \leq p \leq t'$, then

$$\left(\int_{R < |x| < 2R} |K_N(x - y) - K_N(x)|^p dx \right)^{1/p} < CR^{-d+n/p-n/t'} |y|^{d-n/t}$$

for all $|y| < \frac{R}{2}$,

with C independent of N , R , and y .

PROOF. Since $K_N(x) = \sum_{j=-N}^N k_j(x)$,

$$\begin{aligned} & \left(\int_{R < |x| < 2R} |K_N(x - y) - K_N(x)|^p dx \right)^{1/p} \\ & \leq \sum_j \left(\int_{R < |x| < 2R} |k_j(x - y) - k_j(x)|^p dx \right)^{1/p}. \end{aligned} \quad (2.1)$$

Also, $|y| < R/2$ and $R < |x| < 2R$ imply $R/2 < |x - y| < 5R/2$, so that

$$\begin{aligned} & \left(\int_{R < |x| < 2R} |k_j(x - y) - k_j(x)|^p dx \right)^{1/p} \\ & \leq \left(\int_{R < |x| < 2R} |k_j(x - y)|^p dx \right)^{1/p} + \left(\int_{R < |x| < 2R} |k_j(x)|^p dx \right)^{1/p} \\ & \leq 2 \left(\int_{R/2 < |x| < 5R/2} |k_j(x)|^p dx \right)^{1/p}. \end{aligned}$$

Therefore, we need to estimate

$$\left(\int_{R/2 < |x| < 5R/2} |k_j(x)|^p dx \right)^{1/p} \quad \text{and} \quad \left(\int_{R < |x| < 2R} |k_j(x - y) - k_j(x)|^p dx \right)^{1/p}.$$

Let d be an integer such that $0 < d \leq l$ and $1 < t \leq s$ such that $p < t'$. It is easy to see that $m \in M(t, d)$. Let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Then

$$\begin{aligned} & \left(\int_{R/2 < |x| < 5R/2} |k_j(x)|^p dx \right)^{1/p} < CR^{-d} \left(\int_{R/2 < |x| < 5R/2} |x|^d |k_j(x)|^p dx \right)^{1/p} \\ & < CR^{-d} \sum_{|\alpha|=d} \left(\int_{R/2 < |x| < 5R/2} |x^\alpha k_j(x)|^p dx \right)^{1/p}. \end{aligned}$$

Using the fact that $\tilde{m}_j = k_j$, Hölder's inequality, and the Hausdorff-Young

theorem, we have for $|\alpha| = d$ that

$$\begin{aligned}
 \left(\int_{R/2 < |x| < 5R/2} |x^\alpha k_j(x)|^p dx \right)^{1/p} &= \left(\int_{R/2 < |x| < 5R/2} |(D^\alpha m_j)^\vee(x)|^p dx \right)^{1/p} \\
 &= R^{n/p} \left(R^{-n} \int_{R/2 < |x| < 5R/2} |(D^\alpha m_j)^\vee(x)|^p dx \right)^{1/p} \\
 &\leq CR^{n/p} \left(R^{-n} \int_{R/2 < |x| < 5R/2} |(D^\alpha m_j)^\vee(x)|^{t'} dx \right)^{1/t'} \\
 &\leq CR^{n/p-n/t'} \left(\int_{\mathbb{R}^n} |D^\alpha m_j(x)|^t dx \right)^{1/t} \\
 &\leq CR^{n/p-n/t'} (2^j)^{n/t-d}.
 \end{aligned}$$

Combining these estimates gives

$$\left(\int_{R/2 < |x| < 5R/2} |k_j(x)|^p dx \right)^{1/p} \leq CR^{-d+n/p-n/t'} (2^j)^{n/t-d}. \quad (2.2)$$

For the integral of the difference of the k_j 's we have

$$\begin{aligned}
 &\left(\int_{R < |x| < 2R} |k_j(x-y) - k_j(x)|^p dx \right)^{1/p} \\
 &= \left(\int_{R < |x| < 2R} |\{m_j(x)(e^{ix \cdot y} - 1)\}^\vee|^p dx \right)^{1/p} \\
 &\leq CR^{-d} \left(\int_{R < |x| < 2R} |x|^d |\{m_j(x)(e^{ix \cdot y} - 1)\}^\vee|^p dx \right)^{1/p} \\
 &\leq CR^{n/p-d} \sum_{|\alpha|=d} \left(R^{-n} \int_{R < |x| < 2R} |x^\alpha \{m_j(x)(e^{ix \cdot y} - 1)\}^\vee|^p dx \right)^{1/p} \\
 &\leq CR^{n/p-d} \sum_{|\alpha|=d} \left(R^{-n} \int_{R < |x| < 2R} |\{D^\alpha [m_j(x)(e^{ix \cdot y} - 1)]\}^\vee|^p dx \right)^{1/p} \\
 &\leq CR^{n/p-d} \sum_{|\alpha|=d} \left(R^{-n} \int_{R < |x| < 2R} |\{D^\alpha [m_j(x)(e^{ix \cdot y} - 1)]\}^\vee|^{t'} dx \right)^{1/t'} \\
 &\leq CR^{n/p-d-n/t'} \sum_{|\alpha|=d} \left(\int_{\mathbb{R}^n} |D^\alpha [m_j(x)(e^{ix \cdot y} - 1)]|^t dx \right)^{1/t} \\
 &\leq CR^{n/p-d-n/t'} \sum_{|\beta|+|\gamma|=d} \left(\int_{\mathbb{R}^n} |D^\beta m_j(x) \cdot D^\gamma (e^{ix \cdot y} - 1)|^t dx \right)^{1/t}.
 \end{aligned}$$

Consider first $|\gamma| = 0, |\beta| = d$. Since $|e^{ix \cdot y} - 1| \leq |x| |y|$,

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |(D^\beta m_j(x))(e^{ix \cdot y} - 1)|^t dx \right)^{1/t} &\leq \left(\int_{\mathbb{R}^n} |x| |y| |D^\beta m_j(x)|^t dx \right)^{1/t} \\ &\leq C 2^j |y| (2^j)^{n/t-d} = C |y| (2^j)^{n/t-d+1}. \end{aligned}$$

If $|\gamma| > 0, |D^\gamma(e^{ix \cdot y} - 1)| \leq |y|^{|\gamma|}$ and $|\beta| = d - |\gamma|$, so that

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |D^\beta m_j(x) \cdot D^\gamma(e^{ix \cdot y} - 1)|^t dx \right)^{1/t} &\leq \left(\int_{\mathbb{R}^n} |y|^{|\gamma|} |D^\beta m_j(x)|^t dx \right)^{1/t} \\ &\leq C |y|^{|\gamma|} (2^j)^{n/t-|\beta|} = C |y|^{|\gamma|} (2^j)^{n/t-d+|\gamma|}. \end{aligned}$$

Adding these estimates, we obtain

$$\left(\int_{R < |x| < 2R} |k_j(x-y) - k_j(x)|^p dx \right)^{1/p} \leq C R^{n/p-d-n/t'} \sum_{m=1}^d |y|^m (2^j)^{n/t-d+m}. \quad (2.3)$$

But, if $2^j \leq |y|^{-1} (|y| \leq 2^{-j})$,

$$|y|^m (2^j)^{n/t-d+m} \leq |y| (2^j)^{n/t-d+1},$$

so for these values of j , the estimate (2.3) becomes $C R^{n/p-d-n/t'} |y| (2^j)^{n/t-d+1}$.

Using (2.2) and (2.3) in (2.1), we get

$$\begin{aligned} \left(\int_{R < |x| < 2R} |K_N(x-y) - K_N(x)|^p dx \right)^{1/p} &\leq C \sum_{2^j \leq |y|^{-1}} R^{n/p-d-n/t'} |y| (2^j)^{n/t-d+1} + C \sum_{2^j > |y|^{-1}} R^{n/p-d-n/t'} (2^j)^{n/t} \\ &\leq C R^{n/p-d-n/t'} |y|^{d-n/t} \end{aligned}$$

as long as $n/t < d < n/t + 1$. This completes the proof of Lemma 1.

Although we will not use it, we would like to point out that if $l > \max\{n/p', n/s\}$, then

$$\left(\int_{R < |x| < 2R} |K_N(x)|^p dx \right)^{1/p} \leq C R^{n/p-n}.$$

This follows from (2.2) with $d = l$ and the estimate

$$\left(\int_{R < |x| < 2R} |k_j(x)|^p dx \right)^{1/p} \leq C 2^{jn} R^{n/p},$$

which is a consequence of $|k_j(x)| = |\check{m}_j(x)| \leq \|m_j\|_1 \leq C 2^{jn}$.

REMARK 1. We may replace the domain of integration in Lemma 1 by $\{x \in \mathbb{R}^n: R < |x|\}$; that is, under the conditions of Lemma 1,

$$\left(\int_{R < |x|} |K_N(x-y) - K_N(x)|^p dx \right)^{1/p} \leq CR^{-d+n/p-n/t'} |y|^{d-n/t'}.$$

For, if t , d , and y satisfy the conditions of Lemma 1,

$$\begin{aligned} & \left(\int_{R < |x|} |K_N(x-y) - K_N(x)|^p dx \right)^{1/p} \\ & \leq \sum_{j=0}^{\infty} \left(\int_{2^j R < |x| < 2^{j+1} R} |K_N(x-y) - K_N(x)|^p dx \right)^{1/p} \\ & \leq \sum_{j=0}^{\infty} C(2^j R)^{-d+n/p-n/t'} |y|^{d-n/t'} \\ & = CR^{-d+n/p-n/t'} |y|^{d-n/t'} \sum_{j=0}^{\infty} (2^j)^{-d+n/p-n/t'} \\ & = CR^{-d+n/p-n/t'} |y|^{d-n/t'}, \end{aligned}$$

since $-d + n/p - n/t' < 0$ for $n/t < d$.

REMARK 2. The Hörmander-Mikhlin theorem follows easily from Lemma 1. To see this, let $m \in M(s, l)$, $1 < s < 2$ and $l > n/s$. Choose $t < s$ so that $n/t < l < n/t + 1$. By Remark 1 with $p = 1$ and $R = 2|y|$, we have

$$\int_{|x| > 2|y|} |K_N(x-y) - K_N(x)| dx \leq C(2|y|)^{-l+n-n/t'} |y|^{l-n/t'} = C.$$

Thus, the kernels K_N satisfy, uniformly in N , the Hörmander condition

$$\int_{|x| > 2|y|} |K(x-y) - K(x)| dx \leq C \quad \text{for all } y \neq 0,$$

so that $T_N f = K_N * f$ is bounded on L^p , uniformly in N , for $1 < p < \infty$.

For $f \in \mathcal{S}$, we have $Tf = (m\hat{f})^\vee$. It follows that

$$\|Tf - T_N f\|_\infty \leq \|(m - m^N)\hat{f}\|_1 \rightarrow 0$$

since m^N converges pointwise and boundedly to m . Then, applying Fatou's lemma, we get

$$\|Tf\|_p \leq C\|f\|_p,$$

for $f \in \mathcal{S}$, where C is the uniform bound for the T_N on L^p . The result extends to all of L^p by continuity.

Part (1) of Theorem 1 is proved using Lemma 1 and the following three known results.

LEMMA 2. Set $f_r^*(x) = ((f^r)^*)^{1/r}(x)$. If $0 < r < p < \infty$ and $w \in A_{p/r}$, then

$$\|f_r^*\|_{p,w} \leq C\|f\|_{p,w}$$

with C independent of f .

This is an immediate corollary of results in [13].

LEMMA 3. *Let*

$$f^\#(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y) - \text{av}_Q f| dy,$$

where $\text{av}_Q f = |Q|^{-1} \int_Q f(z) dz$. Let $0 < p < \infty$ and $w \in A_\infty$. Then

$$\|f^\#\|_{p,w} < C \|f\|_{p,w}$$

with C independent of f .

This is proved in [4]. The following result is a special case of interpolation with change of measures. It is proved in [17] and [19].

LEMMA 4. *Let $1 < r < q < \infty$ and let w_0 and w_1 be two positive weights. If T is a bounded linear operator from $L_{w_0}^r$ into itself and $L_{w_1}^q$ into itself, then T is bounded from L_w^p into itself for $r < p < q$ and $w = w_0^t w_1^{1-t}$, provided $t = (q-p)/(q-r)$ for $r \neq q$ and $0 < t < 1$ for $r = q$.*

We would like to point out that $w^{n/l} \in A_p$, $n/l > 1$, if and only if $w \in A_p$ and satisfies the reverse Hölder's inequalities

$$\frac{1}{|Q|} \int_Q w^{n/l}(x) dx < C \left(\frac{1}{|Q|} \int_Q w(x) dx \right)^{n/l}$$

and

$$\frac{1}{|Q|} \int_Q (w(x)^{-1/(p-1)})^{n/l} dx < C \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{n/l};$$

when $p = 1$, we only need the first inequality. For $p > 1$, if $w \in A_p$ and satisfies the above inequalities, then

$$\begin{aligned} & \left(\frac{1}{|Q|} \int_Q w^{n/l}(x) dx \right) \left(\frac{1}{|Q|} \int_Q (w^{n/l}(x))^{-1/(p-1)} dx \right)^{p-1} \\ & < C \left(\frac{1}{|Q|} \int_Q w(x) dx \right)^{n/l} \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{(p-1)n/l} \end{aligned}$$

so that $w^{n/l} \in A_p$. For $p = 1$, if $w \in A_1$ and satisfies the first inequality above, then

$$\begin{aligned} \frac{1}{|Q|} \int_Q w^{n/l}(x) dx & < C \left(\frac{1}{|Q|} \int_Q w(x) dx \right)^{n/l} \\ & < C \left(\text{ess inf}_Q w \right)^{n/l} = C \text{ess inf}_Q w^{n/l}, \end{aligned}$$

so that $w^{n/l} \in A_1$. For the other implication, note first that $w^{n/l} \in A_p$ implies $w \in A_p$ since $n/l > 1$. If $p > 1$, by the A_p condition,

$$\left(\frac{1}{|Q|} \int_Q w^{n/l}(x) dx \right)^{l/n} < C \left(\frac{1}{|Q|} \int_Q w(x)^{-(n/l)(1/(p-1))} dx \right)^{-(l/n)(p-1)}.$$

Thus, the first reverse Hölder's inequality will follow if we show

$$\left(\frac{1}{|Q|} \int_Q w(x)^{-(n/l)(1/(p-1))} dx \right)^{-(l/n)(p-1)} < \frac{1}{|Q|} \int_Q w(x) dx,$$

or equivalently

$$1 < \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-n/(p-1)} dx \right)^{(p-1)/n}.$$

But, if $s > 1$, using Hölder's inequality, we have

$$\begin{aligned} 1 &= \frac{1}{|Q|} \int_Q dx = \frac{1}{|Q|} \int_Q w^{1/s}(x) w^{-1/s}(x) dx \\ &< \left(\frac{1}{|Q|} \int_Q w(x) dx \right)^{1/s} \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(s-1)} dx \right)^{(s-1)/s}. \end{aligned}$$

Setting $s - 1 = l(p - 1)/n$, or $s = 1 + l(p - 1)/n > 1$, we get the desired inequality. Since $w^{n/l} \in A_p$ implies $(w^{-1/(p-1)})^{n/l} \in A_{p'}$, we also obtain the other reverse Hölder's inequality from the argument above. Finally, when $p = 1$, by the A_1 condition

$$\frac{1}{|Q|} \int_Q w^{n/l}(x) dx < c \operatorname{ess\,inf}_Q w^{n/l} = c(\operatorname{ess\,inf} w)^{n/l} < c \left(\frac{1}{|Q|} \int_Q w(x) dx \right)^{n/l}.$$

Notice that the above is true if we replace n/l by any $t > 1$.

3. We begin the proof of Theorem 1 by noting that (2) is a consequence of (1) by duality. To see this, suppose $1 < p < (n/l)'$ and $w^{-1/(p-1)} \in A_{p'/n}$. Then, for $f \in \mathfrak{S}$,

$$\|Tf\|_{p,w} = \left(\int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \right)^{1/p} = \sup \left| \int_{\mathbb{R}^n} Tf(x) g(x) dx \right|,$$

where the supremum is taken over all functions $g \in \mathfrak{S}$ such that $\|g\|_{p',w^{-1/(p-1)}} = 1$.

Let $\bar{T}$ be the operator with multiplier $\bar{m}$, the complex conjugate of m . Then $\bar{m}$ satisfies the same estimates as m and we have

$$\begin{aligned} \|Tf\|_{p,w} &= \sup \left| \int_{\mathbb{R}^n} f(x) \bar{T}g(x) dx \right| < \sup \|f\|_{p,w} \|\bar{T}g\|_{p',w^{-1/(p-1)}} \\ &< C \|f\|_{p,w} \sup \|g\|_{p',w^{-1/(p-1)}} = C \|f\|_{p,w} \end{aligned}$$

by (1), since $p' > n/l$ and $w^{-1/(p-1)} \in A_{p'/n}$.

Turning to the proof of (1), fix $p > n/l$ and $w \in A_{p/l/n}$. Choose an $r < s$ such that n/r is not an integer, $n/l < r < p$ and $w \in A_{p/r}$. There is an

integer $d < l$ for which $n/r < d < n/r + 1$. We will show

$$(T_N f)^*(x) < C f_r^*(x) \quad (3.1)$$

with a C independent of f and N .

Fix $x \in \mathbb{R}^n$ and let Q be a cube centered at x with diameter δ . Write

$$f(y) = f_0(y) + \sum_{j=1}^{\infty} f_j(y),$$

where

$$f_0(y) = f(y) \chi(\{y \in \mathbb{R}^n: |x - y| < 2\delta\})$$

and

$$f_j(y) = f(y) \chi(\{y \in \mathbb{R}^n: 2^j \delta < |x - y| < 2^{j+1} \delta\}), \quad j = 1, 2, \dots$$

For $y \in Q$,

$$(K_N * f)(y) = (K_N * f_0)(y) + \sum_{j=1}^{\infty} (K_N * f_j)(y).$$

By Hölder's inequality and Remark 2, for any $q > 1$ we have

$$\begin{aligned} \frac{1}{|Q|} \int_Q |(K_N * f_0)(y)| dy &< \left(\frac{1}{|Q|} \int_Q |(K_N * f_0)(y)|^q dy \right)^{1/q} \\ &< C \frac{\|f_0\|_q}{|Q|^{1/q}} < C f_q^*(x), \end{aligned}$$

with C independent of N . For any j ,

$$\begin{aligned} (K_N * f_j)(y) &= (K_N * f_j)(x) + \int \{K_N(y - z) - K_N(x - z)\} f_j(z) dz \\ &\equiv c_j + e_j, \end{aligned}$$

say. Note that c_j is independent of y and

$$\begin{aligned} |e_j| &< \int_{2^j \delta < |x - z| < 2^{j+1} \delta} |K_N(y - z) - K_N(x - z)| |f(z)| dz \\ &< \left(\int_{2^j \delta < |x - z| < 2^{j+1} \delta} |K_N(y - z) - K_N(x - z)|^{r'} dz \right)^{1/r'} \\ &\quad \cdot \left(\int_{|x - z| < 2^{j+1} \delta} |f(z)|^r dz \right)^{1/r}. \end{aligned}$$

Applying Lemma 1 with $p = r'$ and $t = r$ and noting that $|x - y| < \delta$, we obtain

$$\begin{aligned} |e_j| &< C |x - y|^{d - n/r} (2^j \delta)^{-d} (2^{j+1} \delta)^{n/r} \left\{ (2^{j+1} \delta)^{-n} \int_{|x - z| < 2^{j+1} \delta} |f(z)|^r dz \right\}^{1/r} \\ &< C (2^j)^{n/r - d} f_r^*(x). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{|Q|} \int_Q \left| (K_N * f)(y) - \sum_{j=1}^{\infty} c_j \right| dy &= \frac{1}{|Q|} \int_Q \left| \sum_{j=0}^{\infty} (K_N * f_j)(y) - \sum_{j=1}^{\infty} c_j \right| dy \\ &\leq \frac{1}{|Q|} \int_Q |(K_N * f_0)(y)| dy + \sum_{j=1}^{\infty} \frac{1}{|Q|} \int_Q |(K_N * f_j)(y) - c_j| dy \\ &\leq C f_r^*(x) + C \sum_{j=1}^{\infty} (2^j)^{n/r-d} f_r^*(x) = C f_r^*(x), \end{aligned}$$

since $n/r - d < 0$. The fact that this estimate is true for any cube centered at x implies (3.1). Now, using Lemmas 2 and 3, since $w \in A_{p/r}$, we obtain

$$\|(K_N * f)\|_{p,w} \leq \|(K_N * f)^*\|_{p,w} \leq C \|(K_N * f)^{\sharp}\|_{p,w} \leq C \|f_r^*\|_{p,w} \leq C \|f\|_{p,w},$$

uniformly in N . Arguing as in Remark 2, we have

$$\|Tf\|_{p,w} = \|(K * f)\|_{p,w} \leq C \|f\|_{p,w}.$$

When $l < n$ and $p = n/l$, the above proof fails. However, using Lemma 4 and the fact that $w \in A_1$ implies there is a $b > 1$ such that $w^b \in A_1$, we will prove the result. So, fix such a b . Then $w^b \in A_{q/l/n}$ for any $q > n/l$. Setting $w_0(x) = 1$ and $w_1(x) = w^b(x)$, we need to find q and r so that $r < n/l < q$ and $w(x) = (w^b(x))^{(n/l-r)/(q-r)}$. Thus we need $b((n/l-r)/(q-r)) = 1$ or $b(n/l-r) = q-r$. Then, choosing r , $1 < r < n/l$, and solving for q , which is necessarily greater than n/l since $b > 1$, completes the proof.

The proof of Theorem 1 will be finished once we show the weak-type (1, 1) result. This will be done using standard techniques which are included for completeness. Fix a nonnegative f in $L^1 \cap L_w^1$ and $\lambda > 0$. Applying the Calderón-Zygmund decomposition to f , we get a sequence of disjoint cubes $\{Q_k\}$ and functions g and b , $f(x) = g(x) + b(x)$, satisfying

$$(i) |Q_k| \leq (C/\lambda) \int_{Q_k} f(y) dy,$$

$$(ii) \|g\|_{2,w}^2 \leq \lambda \|f\|_{1,w},$$

$$(iii) b(y) = f(y) - |Q_k|^{-1} \int_{Q_k} f(z) dz \text{ for } y \in Q_k, \text{ supp } b \subset \bigcup Q_k \text{ and } \int_{Q_k} b(y) dy = 0.$$

Since $T_N f = T_N g + T_N b$,

$$m_w(\{x \in \mathbb{R}^n: |T_N f(x)| > 2\lambda\})$$

$$\leq m_w(\{x \in \mathbb{R}^n: |T_N g(x)| > \lambda\}) + m_w(\{x \in \mathbb{R}^n: |T_N b(x)| > \lambda\}).$$

We can apply (i) of Theorem 1 to the first term on the right because $w \in A_1$. Then, using (ii), we get

$$m_w(\{x \in \mathbb{R}^n: |T_N g(x)| > \lambda\}) \leq \frac{C}{\lambda^2} \|g\|_{2,w}^2 \leq \frac{C}{\lambda} \|f\|_{1,w}.$$

Let Q_k^* be Q_k expanded concentrically twice. Then using (i) and the fact that

$w \in A_1$, we have

$$\begin{aligned} m_w\left(\bigcup Q_k^*\right) &< \sum m_w(Q_k^*) < C \sum m_w(Q_k) < C \sum \frac{1}{\lambda} \int_{Q_k} f(y) \frac{m_w(Q_k)}{|Q_k|} dy \\ &< \frac{C}{\lambda} \sum \int_{Q_k} f(y) w(y) dy < \frac{C}{\lambda} \|f\|_{1,w}. \end{aligned}$$

Thus, we have only to show

$$m_w\left(\left\{x \notin \bigcup Q_k^*: |T_N b(x)| > \lambda\right\}\right) < \frac{C}{\lambda} \|f\|_{1,w}. \quad (3.2)$$

Let y_k and δ_k be the center and diameter of Q_k . Then

$$\begin{aligned} \int_{x \notin \bigcup Q_k^*} |T_N b(x)| w(x) dx &= \int_{x \notin \bigcup Q_k^*} \left| \int_{\mathbb{R}^n} K_N(x-y) b(y) dy \right| w(x) dx \\ &= \int_{x \notin \bigcup Q_k^*} \left| \sum_k \int_{Q_k} K_N(x-y) b(y) dy \right| w(x) dx \\ &= \int_{x \notin \bigcup Q_k^*} \left| \sum_k \int_{Q_k} \{K_N(x-y) - K_N(x-y_k)\} b(y) dy \right| w(x) dx \\ &< \sum_k \int_{Q_k} \left(\int_{x \notin Q_k^*} |K_N(x-y) - K_N(x-y_k)| w(x) dx \right) |b(y)| dy. \end{aligned}$$

If we can show, for any $y \in Q_k$, that the inner integral is bounded by a constant independent of k and N times $\text{ess inf}_{Q_k} w$, then our result will follow, as we now show. For, by (iii),

$$\begin{aligned} m_w\left(\left\{x \notin \bigcup Q_k^*: |T_N b(x)| > \lambda\right\}\right) &< \frac{1}{\lambda} \int_{x \notin \bigcup Q_k^*} |T_N b(x)| w(x) dx \\ &< \frac{C}{\lambda} \sum \int_{Q_k} |b(x)| \text{ess inf}_{Q_k} w dx < \frac{C}{\lambda} \sum \int_{Q_k} |b(x)| w(x) dx \\ &< \frac{C}{\lambda} \sum \int_{Q_k} f(x) w(x) dx + \frac{C}{\lambda} \sum \int_{Q_k} \left(\frac{1}{|Q_k|} \int_{Q_k} f(z) dz \right) w(x) dx \\ &< \frac{C}{\lambda} \|f\|_{1,w} + \frac{C}{\lambda} \sum \int_{Q_k} f(z) \frac{m_w(Q_k)}{|Q_k|} dz \\ &< \frac{C}{\lambda} \|f\|_{1,w} + \frac{C}{\lambda} \sum \int_{Q_k} f(z) w(z) dz < \frac{2C}{\lambda} \|f\|_{1,w}. \end{aligned}$$

Therefore,

$$m_w\left(\left\{x \in \mathbb{R}^n: |T_N f(x)| > \lambda\right\}\right) < \frac{C}{\lambda} \|f\|_{1,w} \quad (3.3)$$

with a constant independent of N , f , and λ . If $f \in \mathcal{S}$, $f = f^+ - f^-$ where f^+ and f^- are nonnegative and in $L^1 \cap L^1_w$, so that (3.3) holds for $f \in \mathcal{S}$. Then

$$\begin{aligned}
& m_w(\{x \in \mathbb{R}^n: |Tf(x)| > \lambda\}) \\
& \leq m_w(\{x \in \mathbb{R}^n: |T_N f(x)| + |Tf(x) - T_N f(x)| > \lambda\}) \\
& \leq m_w\left(\left\{x \in \mathbb{R}^n: |T_N f(x)| > \frac{\lambda}{2}\right\}\right) \\
& \quad + m_w\left(\left\{x \in \mathbb{R}^n: |Tf(x) - T_N f(x)| > \frac{\lambda}{2}\right\}\right).
\end{aligned}$$

Since $T_N f$ converges uniformly to Tf for $f \in \mathfrak{S}$, choosing N large enough the second term on the right is zero. By (3.3),

$$m_w(\{x \in \mathbb{R}^n: |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{1,w} \quad \text{for } f \in \mathfrak{S},$$

which extends to L_w^1 .

To complete the proof of Theorem 1 we need to show

$$\int_{x \notin Q_k^*} |K_N(x-y) - K_N(x-y_k)| w(x) dx \leq C \operatorname{ess\,inf}_{Q_k} w \quad \text{if } y \in Q_k,$$

with C independent of k and N . Choose $r < s$ so that $n/r < l < n/r + 1$ and $w^r \in A_1$. Then, using Lemma 1 with $p = r'$ and $t = r$ and noting that $x \notin Q_k^*$ implies $|x - y_k| > 2\delta_k$, we have for $y \in Q_k$ that

$$\begin{aligned}
& \int_{|x-y_k| > 2\delta_k} |K_N(x-y) - K_N(x-y_k)| w(x) dx \\
& = \sum_{j=1}^{\infty} \int_{2^j \delta_k < |x-y_k| < 2^{j+1} \delta_k} |K_N(x-y) - K_N(x-y_k)| w(x) dx \\
& \leq \sum_{j=1}^{\infty} \left(\int_{2^j \delta_k < |x-y_k| < 2^{j+1} \delta_k} |K_N(x-y) - K_N(x-y_k)|^{r'} dx \right)^{1/r'} \\
& \quad \cdot \left(\int_{|x-y_k| < 2^{j+1} \delta_k} w^r(x) dx \right)^{1/r} \\
& \leq C \sum_{j=1}^{\infty} (2^j \delta_k)^{-l} (\delta_k)^{l-n/r} (2^{j+1} \delta_k)^{n/r} \left\{ (2^{j+1} \delta_k)^{-n} \int_{|x-y_k| < 2^{j+1} \delta_k} w^r(x) dx \right\}^{1/r}.
\end{aligned}$$

Thus, since $w^r \in A_1$,

$$\begin{aligned}
& \int_{|x-y_k| > 2\delta_k} |K_N(x-y) - K_N(x-y_k)| w(x) dx \\
& \leq C \sum_{j=1}^{\infty} (2^j)^{n/r-l} \operatorname{ess\,inf}_{|x-y_k| < 2^{j+1} \delta_k} w(x) \\
& \leq C \operatorname{ess\,inf}_{|x-y_k| < \delta_k} w(x) \sum_{j=1}^{\infty} (2^j)^{n/r-l} \leq C \operatorname{ess\,inf}_{Q_k} w
\end{aligned}$$

with C independent of k and N . This completes the proof of Theorem 1.

We will derive Theorem 2 from Theorem 1 by using Lemma 4 and a characterization of A_p functions proved by P. Jones [9]. He has shown that if $w \in A_p$ then there are A_1 weights u and v such that $w = uv^{1-p}$.

Fix p , $1 < p < \infty$, and w so that $w^{n/l} \in A_p$. We have $w^{n/l} = uv^{1-p}$, $u, v \in A_1$, or $w = u^{l/n} v^{l(1-p)/n}$. Next, write this as

$$w = u^{l/n} v^{l(1-p)/n} = (u^\alpha v^\beta)^t (u^\gamma v^\delta)^{1-t} = w_0^t w_1^{1-t}.$$

For this to make sense, we need

$$\alpha t + \gamma(1-t) = \frac{l}{n}, \quad (3.4)$$

$$\beta t + \delta(1-t) = \frac{l}{n}(1-p). \quad (3.5)$$

Then, in order to use Lemma 4 for weights which satisfy Theorem 1, we require

$$w_0^{-1/(r-1)} \in A_{r'l/n}, \quad 1 < r < \min\left\{\left(\frac{n}{l}\right)', p\right\}, \quad (3.6)$$

$$w_1 \in A_{q'l/n}, \quad q > \max\left\{\frac{n}{l}, p\right\}, \quad (3.7)$$

$$t = \frac{q-p}{q-r}. \quad (3.8)$$

Recall that $u \in A_1$ (similarly $v \in A_1$) implies

$$\frac{1}{|Q|} \int_Q u(y) dy < Cu(x) \quad \text{for almost all } x \in Q.$$

Therefore, if $\alpha > 0$ and $\beta < 0$, letting $s = r'l/n$, we have

$$\begin{aligned} & \left(\frac{1}{|Q|} \int_Q w_0(x)^{-1/(r-1)} dx \right) \left(\frac{1}{|Q|} \int_Q w_0(x)^{(1/(r-1))(1/(s-1))} dx \right)^{s-1} \\ &= \left(\frac{1}{|Q|} \int_Q u(x)^{-\alpha/(r-1)} v(x)^{-\beta/(r-1)} dx \right) \\ & \quad \cdot \left(\frac{1}{|Q|} \int_Q u(x)^{(\alpha/(r-1))(1/(s-1))} v(x)^{(\beta/(r-1))(1/(s-1))} dx \right)^{s-1} \\ &< C \left(\frac{1}{|Q|} \int_Q u(x) dx \right)^{-\alpha/(r-1)} \left(\frac{1}{|Q|} \int_Q v(x)^{-\beta/(r-1)} dx \right) \\ & \quad \cdot \left(\frac{1}{|Q|} \int_Q v(x) dx \right)^{\beta/(r-1)} \left(\frac{1}{|Q|} \int_Q u(x)^{(\alpha/(r-1))(1/(s-1))} dx \right)^{s-1} \\ &= C, \end{aligned}$$

if

$$\alpha = (r-1)\left(\frac{r'l}{n} - 1\right) = \frac{r'l}{n} - r + 1 \quad \text{and} \quad \beta = -(r-1);$$

that is $w_0^{-1/(r-1)} \in A_{r'l/n}$ for these values of α and β . Similarly, we can show $w_1 \in A_{q'l/n}$ if $\gamma = 1$ and $\delta = -((ql/n) - 1)$. Using these values of α and γ , we have (3.4) if $t = 1/r$. Next, solving (3.5) for q , we get $q = r'(p-1)$. This value of q also satisfies (3.8). Therefore, if we choose $r < \min\{(n/l)', p\}$ so close to 1 that $q = r'(p-1) > \max\{n/l, p\}$, we can satisfy (3.4)–(3.8), proving Theorem 2.

Before proving Theorem 3, notice that $-n > -lp$ if $n/l < p$, and $n(p-1) < lp$ if $p < (n/l)'$. Therefore, for $l < n$ the conclusion of Theorem 3 can be divided into three cases:

$$1 < p < \frac{n}{l} \quad \text{and} \quad -lp < \beta < n(p-1), \quad (3.9)$$

$$\frac{n}{l} < p < \left(\frac{n}{l}\right)' \quad \text{and} \quad -n < \beta < n(p-1), \quad (3.10)$$

$$\left(\frac{n}{l}\right)' < p < \infty \quad \text{and} \quad -n < \beta < lp. \quad (3.11)$$

Since (3.11) is the dual of (3.9), we need only concern ourselves with (3.9) and (3.10).

Next, let us interpret Theorem 1 when $w(x)$ is a power of $|x|$. Because $|x|^\beta \in A_p$ if and only if $-n < \beta < n(p-1)$, we have ($l < n$) that T is bounded on $L_{|x|^\beta}^p$ if

$$\frac{n}{l} < p < \infty \quad \text{and} \quad -n < \beta < pl - n, \quad (3.12)$$

$$1 < p < \left(\frac{n}{l}\right)' \quad \text{and} \quad -n + p(n-l) < \beta < n(p-1). \quad (3.13)$$

However, combining (3.12) and (3.13), we have (3.10) and are left with only proving (3.9).

Let $q = n/l$ and $r < n/l$; then also $r < (n/l)'$. By (3.13) and (3.10), T is bounded on $L_{|x|^{\beta_0}}^r$ and $L_{|x|^{\beta_1}}^q$ for $-n + r(n-l) < \beta_0 < n(r-1)$ and $-n < \beta_1 < n(q-1)$. Using Lemma 4, if $r < p < q$ we see that T is bounded on $L_{|x|^\beta}^p$ for

$$\beta = \beta_0 \left(\frac{q-p}{q-r} \right) + \beta_1 \left(\frac{p-r}{q-r} \right).$$

Thus β satisfies

$$\begin{aligned} & \{-n + r(n-l)\} \left(\frac{q-p}{q-r} \right) - n \left(\frac{p-r}{q-r} \right) \\ & < \beta < n(r-1) \left(\frac{q-p}{q-r} \right) + n(q-1) \left(\frac{p-r}{q-r} \right). \end{aligned}$$

Simplifying and using the fact that $q = n/l$, we get

$$\frac{n^2(r-1)}{n-lr} + \frac{plr(l-n)}{n-lr} < \beta < n(p-1). \quad (3.14)$$

But, as $r \rightarrow 1$, the left-hand side of (3.14) approaches $-lp$. So, taking r sufficiently close to 1 allows us to choose any β satisfying $-lp < \beta < n(p-1)$.

When $l = n$, the restriction in Theorem 3 is $-n < \beta < n(p-1)$ for $1 < p < \infty$. But, when $l = n$ in Theorem 1, we require $w \in A_p$, and $|x|^\beta \in A_p$ if $-n < \beta < n(p-1)$.

4. The proof of Theorem 4 is based on an analogue of Lemma 1.

LEMMA 5. Let $\Omega \in L'(\Sigma)$ and satisfy the L^r -Dini condition. Set $K(x) = \Omega(x')/|x|^n$. There exists a constant $\alpha_0 > 0$ such that if $|y| < \alpha_0 R$, then

$$\left(\int_{R < |x| < 2R} |K(x-y) - K(x)|^r dx \right)^{1/r} \\ < CR^{n/r-n} \left\{ \frac{|y|}{R} + \int_{|y|/2R < \delta < |y|/R} \omega_r(\delta) \frac{d\delta}{\delta} \right\}.$$

PROOF. We may choose $\alpha_0 < \frac{1}{2}$; then, since $|x| > R$, $|x-y|$ is equivalent to $|x|$. Therefore,

$$|K(x-y) - K(x)| = \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| \\ < C \left\{ |\Omega(x)| \frac{|y|}{|x|^{n+1}} + \frac{|\Omega(x-y) - \Omega(x)|}{|x|^n} \right\}.$$

It follows that

$$\left(\int_{R < |x| < 2R} |K(x-y) - K(x)|^r dx \right)^{1/r} \\ < C \left(\int_{R < |x| < 2R} |\Omega(x)|^r \frac{|y|^r}{|x|^{(n+1)r}} dx \right)^{1/r} \\ + C \left(\int_{R < |x| < 2R} \frac{|\Omega(x-y) - \Omega(x)|^r}{|x|^{nr}} dx \right)^{1/r}. \quad (4.1)$$

The first term on the right side of (4.1) is bounded by

$$C \|\Omega\|_{L'(\Sigma)} |y| R^{-(n+1)} R^{n/r} = CR^{n/r-n} \left(\frac{|y|}{R} \right).$$

Changing to polar coordinates, we see the second term equals

$$C \left(\int_R^{2R} t^{-nr+n-1} \left(\int_{\Sigma} |\Omega(tx' - y) - \Omega(tx')|^r d\sigma_{x'} \right) dt \right)^{1/r} \\ \leq CR^{n/r-n} \left(\int_R^{2R} \left(\int_{\Sigma} \left| \Omega \left(\frac{x' - \alpha}{|x' - \alpha|} \right) - \Omega(x') \right|^r d\sigma_{x'} \right) \frac{dt}{t} \right)^{1/r},$$

where $\alpha = y/t$. Arguing as in Calderón, Weiss, and Zygmund [1, pp. 65–72], we see the inner integral is bounded by

$$C \sup_{|\rho| \leq |\alpha|} \int_{\Sigma} |\Omega(\rho x') - \Omega(x')|^r d\sigma_{x'} = C\omega_r \left(\frac{|y|}{t} \right)$$

as long as $|\alpha| = |y|/t < \alpha_0$. Thus, the second term is bounded by

$$CR^{n/r-n} \left(\int_R^{2R} \omega_r \left(\frac{|y|}{t} \right) \frac{dt}{t} \right)^{1/r} = CR^{n/r-n} \left(\int_{|y|/2R}^{|y|/R} \omega_r(\delta) \frac{d\delta}{\delta} \right)^{1/r} \\ \leq CR^{n/r-n} \left(\int_{|y|/2R < \delta < |y|/R} \omega_r(\delta) \frac{d\delta}{\delta} \right),$$

since ω_r is essentially constant on intervals of the form $(a, 2a)$, $a > 0$. Lemma 5 is now proved.

Notice that when $R = 2^j|y|$, with a j such that $1/\alpha_0 < 2^j$, we get

$$\left(\int_{2^j|y| < |x| < 2^{j+1}|y|} |K(x - y) - K(x)|^r dx \right)^{1/r} \\ \leq C(2^j|y|)^{n/r-n} \left\{ \frac{1}{2^j} + \int_{2^{-(j+1)}}^{2^{-j}} \omega_r(\delta) \frac{d\delta}{\delta} \right\}.$$

Theorem 4 is proved in exactly the same manner as Theorem 1. Using Lemma 5, we show

$$(K * f)^{\sharp}(x) \leq C f_r^{\sharp}(x),$$

which proves the result for $p > r'$. The only change necessary is in the decomposition $f = f_0 + \sum f_j$. For Theorem 4,

$$f_0(y) = f(y) \chi \left(\left\{ y \in \mathbb{R}^n: |x - y| \leq \frac{1}{\alpha_0} \delta \right\} \right)$$

and the sum of f_j 's is over $j \geq \log_2(1/\alpha_0)$. We get the case $p = r'$ by interpolation, and $1 < p < r$ follows by duality. In the weak-type $(1, 1)$ proof, we may have to replace the weak-type $(2, 2)$ result for the good function by a weak-type (r', r') result.

5. We conclude by showing that Theorem 3 is best possible, except for endpoint equalities for β . We prove the result for $p > (n/l)'$; the case $p < n/l$ follows by duality. For $n/l < p < (n/l)'$, the Riesz transforms and

an argument like that in [8] show the range of β is best possible.

Let $1 < s < 2$, $n/s < l < n$, $(n/l)' < p$ and $\beta > lp$. Define a multiplier m by

$$m(x) = e^{ix \cdot \eta} (1 + |x|^2)^{-l/2}$$

for a fixed η of length 1. Note that $\check{m}(x) = G_l(x - \eta)$ (the Bessel kernel of order l) and that $|D^\alpha m(x)| \leq C_\alpha / (1 + |x|)^l$, so $m \in M(s, l)$, $1 < s < \infty$. Moreover, $G_l \geq 0$ and there exist $c, \mu > 0$ such that $G_l(x) > c|x|^{l-n}$ if $|x| < \mu$ (see Stein [19, p. 132] for details).

Set

$$f(x) = |x|^{-((n+\beta)/p)} |\log|x||^{-\delta} \chi(\{x \in \mathbb{R}^n: |x| < \mu\}).$$

If $\delta p > 1$, $f \in L^p_{|x|^\beta}(\mathbb{R}^n)$. Since $Tf(x) = (G_l(\cdot - \eta) * f)(x)$,

$$\begin{aligned} Tf(x) &= \int_{|y| < \mu} |y|^{-((n+\beta)/p)} |\log|y||^{-\delta} G_l(x - y - \eta) dy \\ &= \int_{|x - \eta - z| < \mu} |x - \eta - z|^{-((n+\beta)/p)} |\log|x - \eta - z||^{-\delta} G_l(z) dz \end{aligned}$$

by setting $z = x - \eta - y$. Now, if we restrict the integration to $|z| < \frac{1}{2}|x - \eta|$, $|x - \eta - z|$ is equivalent to $|x - \eta|$ and, if $|x - \eta| < \mu/2$,

$$\begin{aligned} Tf(x) &\geq C|x - \eta|^{-((n+\beta)/p)} |\log|x - \eta||^{-\delta} \int_{|z| < \frac{1}{2}|x - \eta|} \frac{dz}{|z|^{n-l}} \\ &= C|x - \eta|^{l - ((n+\beta)/p)} |\log|x - \eta||^{-\delta}. \end{aligned}$$

Therefore, $Tf \notin L^p_{|x|^\beta}(\mathbb{R}^n)$ if $\{l - ((n + \beta)/p)\}p < -n$; i.e., if $\beta > lp$.

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903